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## Algebras with inner MB-representation

### Abstract

We investigate algebras of sets, and pairs  $\langle \mathcal{A}, \mathcal{I} \rangle$  consisting of an algebra  $\mathcal{A}$  and an ideal  $\mathcal{I} \subset \mathcal{A}$ , that possess an inner MB-representation. We compare inner MB-representability of  $\langle \mathcal{A}, \mathcal{I} \rangle$  with several properties of  $\langle \mathcal{A}, \mathcal{I} \rangle$  considered by Baldwin. We show that  $\mathcal{A}$  is inner MB-representable if and only if  $\mathcal{A} = S(\mathcal{A} \setminus \mathcal{H}(\mathcal{A}))$ , where  $S(\cdot)$  is a Marczewski operation defined below and  $\mathcal{H}$  consists of sets that are hereditarily in  $\mathcal{A}$ . We study the question of uniqueness of the ideal in that representation.

## 1 The implications

Let  $X$  be a nonempty set and let  $\mathcal{F}$  be a nonempty family of nonempty subsets of  $X$ . Following the idea of Burstin and Marczewski we define:

$$S(\mathcal{F}) = \{A \subset X : (\forall P \in \mathcal{F})(\exists Q \in \mathcal{F})(Q \subset A \cap P \text{ or } Q \subset P \setminus A)\}$$

and

$$S_0(\mathcal{F}) = \{A \subset X : (\forall P \in \mathcal{F})(\exists Q \in \mathcal{F})(Q \subset P \setminus A)\}.$$

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Then  $S(\mathcal{F})$  is an algebra of subsets of  $X$  and  $S_0(\mathcal{F})$  is an ideal on  $X$ . (See [BBRW].) For an ideal  $\mathcal{I}$  on  $X$  an algebra  $\mathcal{A}$  of subsets of  $X$  such that  $\mathcal{I} \subset \mathcal{A}$  we say that

- the pair  $\langle \mathcal{A}, \mathcal{I} \rangle$  (respectively, the algebra  $\mathcal{A}$ ) *has inner MB-representation* provided there exists an  $\mathcal{F} \subset \mathcal{A}$  such that  $\mathcal{A} = S(\mathcal{F})$  and  $\mathcal{I} = S_0(\mathcal{F})$  (respectively,  $\mathcal{A} = S(\mathcal{F})$ );
- the pair  $\langle \mathcal{A}, \mathcal{I} \rangle$  *has density property* provided  $\mathcal{I} = S_0(\mathcal{A} \setminus \mathcal{I})$ ;
- the pair  $\langle \mathcal{A}, \mathcal{I} \rangle$  (respectively, the algebra  $\mathcal{A}$ ) *is topological* provided there exists a topology  $\tau$  on  $X$  such that  $\langle \mathcal{A}, \mathcal{I} \rangle = \langle S(\mathcal{F}), S_0(\mathcal{F}) \rangle$  (respectively,  $\mathcal{A} = S(\mathcal{F})$ ), where  $\mathcal{F} = \tau \setminus \{\emptyset\}$ ;
- the pair  $\langle \mathcal{A}, \mathcal{I} \rangle$  *has the hull property* provided for every  $U \subset X$  there is a  $V \in \mathcal{A}$  such that  $U \subset V$  and for every  $W \in \mathcal{A}$  if  $U \subset W$  then  $V \setminus W \in \mathcal{I}$ ;
- the pair  $\langle \mathcal{A}, \mathcal{I} \rangle$  *is complete* provided the quotient algebra  $\mathcal{A}/\mathcal{I}$  is complete;
- the pair  $\langle \mathcal{A}, \mathcal{I} \rangle$  *has the splitting property* provided for every  $\mathcal{C} \subset \mathcal{D} \subset \mathcal{A}$ , if  $\mathcal{D}$  is an antichain (i.e.,  $A \cap B \in \mathcal{I}$  for every distinct  $A, B \in \mathcal{D}$ ) then there exists a mapping  $\mathcal{D} \ni D \mapsto I_D \in \mathcal{I}$  such that  $C \setminus I_C$  and  $D \setminus I_D$  are disjoint for every  $C \in \mathcal{C}$  and  $D \in \mathcal{D} \setminus \mathcal{C}$ .

In the graph from Theorem 2 each of these properties is denoted, respectively, as: inner, dense, top, hull, comp, and split.

We start here with the following simple characterization of pairs with inner MB-representation. (Compare also [Wr, lemma 1].)

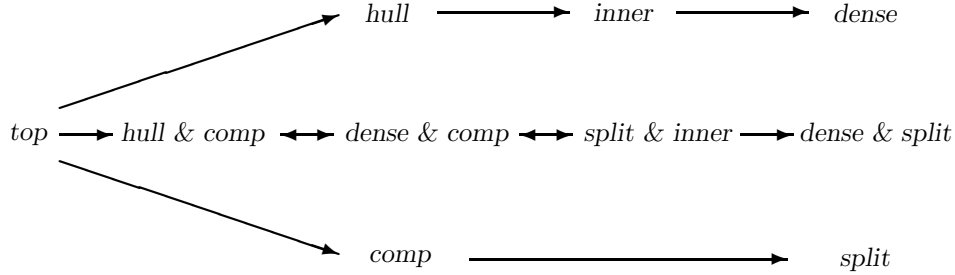
**Proposition 1** *A pair  $\langle \mathcal{A}, \mathcal{I} \rangle$  has an inner MB-representation if and only if  $\mathcal{A} = S(\mathcal{A} \setminus \mathcal{I})$ .*

PROOF. If  $\mathcal{A} = S(\mathcal{A} \setminus \mathcal{I})$  then  $\mathcal{A} \setminus \mathcal{I} \subset \mathcal{A} \setminus S_0(\mathcal{A} \setminus \mathcal{I})$ , since we always have  $\mathcal{F} \cap S_0(\mathcal{F}) = \emptyset$ . So,  $S_0(\mathcal{A} \setminus \mathcal{I}) \subset \mathcal{I}$ . The other inclusion is obvious. Thus,  $\langle \mathcal{A}, \mathcal{I} \rangle$  has an inner MB-representation.

Conversely, assume that  $\langle \mathcal{A}, \mathcal{I} \rangle = \langle S(\mathcal{F}), S_0(\mathcal{F}) \rangle$  for some  $\mathcal{F} \subset \mathcal{A}$ . By [BBRW, prop. 1.2] to prove that  $S(\mathcal{A} \setminus \mathcal{I}) = S(\mathcal{F})$  it is enough to show that the families  $\mathcal{A} \setminus \mathcal{I}$  and  $\mathcal{F}$  are *mutually coinital*, that is, every element of each of these families contains an element from the other.

Clearly,  $\mathcal{F} \subset \mathcal{A} \setminus S_0(\mathcal{F}) = \mathcal{A} \setminus \mathcal{I}$ , so every element of  $\mathcal{F}$  contains an element from  $\mathcal{A} \setminus \mathcal{I}$ . Conversely, if  $A \in \mathcal{A} \setminus \mathcal{I}$  then there exists an  $F \in \mathcal{F}$  with  $F \subset A$ , since  $A \notin \mathcal{I} = S_0(\mathcal{F})$ . ■

**Theorem 2** *We have the following implications between the properties of a pair  $\langle \mathcal{A}, \mathcal{I} \rangle$ .*



Diagram

Moreover, none of the implications can be reversed, with possible exception of “ $\text{top} \Rightarrow \text{hull \& comp}$ .”

PROOF. The facts that every topological pair is complete and has the hull property are well known and easy to see. Indeed, if  $\langle \mathcal{A}, \mathcal{I} \rangle$  is a topological pair generated by a topology  $\tau$  on  $X$  then  $\mathcal{I}$  consists of all nowhere dense sets (with respect to  $\tau$ ) and  $\mathcal{A}$  consists of open sets (with respect to  $\tau$ ) modulo  $\mathcal{I}$ . (See [BR].) Then, for each  $E \subset X$ , the closure  $\text{cl}(E)$  plays a role of its hull. Since an open set  $U$  can be expressed as  $U = V \setminus E$  where  $V$  is regular open and  $E$  is nowhere dense (see e.g. [O, thm. 4.5]), the quotient algebra  $\mathcal{A}/\mathcal{I}$  is isomorphic to the Boolean algebra of regular open sets, which is complete (see e.g. [K]). Hence  $\mathcal{A}/\mathcal{I}$  is complete.

The implication “ $\text{inner} \Rightarrow \text{dense}$ ” results immediately from Proposition 1 and the definitions. All other implications follow from the following implications proved in Baldwin’s paper [Ba]: “ $\text{hull} \Rightarrow \text{inner}$ ,” “ $\text{comp} \Rightarrow \text{split}$ ,” “ $\text{split \& inner} \Rightarrow \text{comp}$ ,” and “ $\text{dense \& comp} \Rightarrow \text{hull}$ .”

The fact that the implications “ $\text{top} \Rightarrow \text{hull}$ ” and “ $\text{top} \Rightarrow \text{comp}$ ” cannot be reversed follows from Baldwin’s examples from [Ba], where he shows that the properties hull and complete are independent of each other.

An example showing that “ $\text{dense \& split}$ ” does not imply “ $\text{inner}$ ” is described in Example 3. This takes care of nonreversability of the implications “ $\text{split \& inner} \Rightarrow \text{dense \& split}$ ,” “ $\text{inner} \Rightarrow \text{dense}$ ,” and “ $\text{comp} \Rightarrow \text{split}$ .”

Example 4 shows that the implications “ $\text{hull} \Rightarrow \text{inner}$ ” cannot be reversed.

■

The following example answers a question of Baldwin [Ba, question 2] whether every pair with density and splitting properties must be inner. Also,

Baldwin had the example of a family with a splitting property which is not complete only under the assumption of the continuum hypothesis, while the example below is in ZFC.

**Example 3** *If  $X$  is an infinite set,  $\mathcal{A}$  is an algebra of subsets of  $X$  which are either finite or cofinite, and  $\mathcal{I} = \{\emptyset\}$  then the pair  $\langle \mathcal{A}, \mathcal{I} \rangle$  has density and splitting properties but is neither inner nor complete.*

PROOF. The pair  $\langle \mathcal{A}, \mathcal{I} \rangle$  has density property since  $S_0(\mathcal{A} \setminus \{\emptyset\}) = \{\emptyset\} = \mathcal{I}$ . It does not have inner MB-representation by Proposition 1 and the fact that  $S(\mathcal{A} \setminus \{\emptyset\}) = \mathcal{P}(X)$ . The splitting property is satisfied trivially, since  $\mathcal{I} = \{\emptyset\}$ .

The pair  $\langle \mathcal{A}, \mathcal{I} \rangle$  is not complete by the implications from Theorem 2. ■

The following example answers a question of Baldwin [Ba, question 1] whether every pair with inner MB-representation must have a hull property.

In what follows we use the standard set theoretic notation as in [Ci]. Let  $X$  be an infinite set of cardinality  $\kappa$ . We say that a family  $\mathcal{F}_0 \subset [X]^\kappa$  is *almost disjoint* provided  $|F_1 \cap F_2| < \kappa$  for every distinct  $F_1, F_2 \in \mathcal{F}_0$ .

**Example 4** *There exists a maximal almost disjoint family  $\mathcal{F}_0 \subset [X]^\kappa$  such that for  $\mathcal{F} = \{F \triangle A : F \in \mathcal{F}_0 \text{ \& } A \in [X]^{<\kappa}\}$  the pair  $\langle S(\mathcal{F}), S_0(\mathcal{F}) \rangle$  has inner MB-representation but neither is complete nor it has the hull property.*

PROOF. In [BC, fact 4] it was proved that for every  $\mathcal{F}$  as in the theorem the algebra  $S(\mathcal{F})$  contains  $\mathcal{F}$  (so it has inner MB-representation) and  $S_0(\mathcal{F}) = [X]^{<\kappa}$ .

Let  $\{A, B\}$  be a partition of  $X$  into the sets of cardinality  $\kappa$  and let  $\mathcal{G} \subset [X]^\kappa$  be a partition of  $X$  into  $\kappa$  many sets such that  $|G \cap A| = |G \cap B| = \kappa$  for every  $G \in \mathcal{G}$ . Let  $\mathcal{F}_0 \subset [X]^\kappa$  be a maximal almost disjoint family extending  $\mathcal{G}$  such that for every  $F \in \mathcal{F}_0$  either  $F \subset A$  or  $F \subset B$ . Such an  $\mathcal{F}_0$  exists by the Zorn lemma. It is easy to see that  $\mathcal{F}_0$  is a maximal almost disjoint family in  $[X]^\kappa$ .

To see that  $\langle S(\mathcal{F}), S_0(\mathcal{F}) \rangle$  does not have the hull property notice that  $A \subset X$  does not have a hull. Indeed, take a  $V \in S(\mathcal{F})$  containing  $A$ . Then for every  $G \in \mathcal{G} \subset \mathcal{F}$  there is an  $F_G \in \mathcal{F}$  contained in  $G$  such that  $F_G$  is either disjoint or contained in  $V$ . Thus,  $F_G = G \setminus A_G$  for some  $A_G \in [X]^{<\kappa}$ , since elements of  $\mathcal{F}_0$  are almost disjoint. This implies also that  $F_G = G \setminus A_G$  must be a subset of  $V$ , since it cannot be disjoint with  $V \supset A$ . In other words, for every  $G \in \mathcal{G}$  there exists an  $x_G \in G \cap (V \setminus A)$ . So,  $Y = \{x_G : G \in \mathcal{G}\} \in [B]^\kappa$ , and by the maximality, there exists an  $F \in \mathcal{F}_0$  such that  $|F \cap Y| = \kappa$ . Then, for  $W = V \setminus F \in S(\mathcal{F})$  we have  $A \subset W \subset V$ , while  $V \setminus W = F \cap Y \notin [X]^{<\kappa} = S_0(\mathcal{F})$ . Thus, there is no hull for  $A$  with respect to  $\langle S(\mathcal{F}), S_0(\mathcal{F}) \rangle$ . ■

**Problem 5** Is every complete pair  $\langle \mathcal{A}, \mathcal{I} \rangle$  with the hull property topological?

## 2 Notes on algebras with inner MB-representations

According to Proposition 1 if a pair  $\langle \mathcal{A}, \mathcal{I} \rangle$  has inner MB-representation then it has a canonical one — by a family  $\mathcal{F} = \mathcal{A} \setminus \mathcal{I}$ . But what if we only consider inner MB-representability of an algebra  $\mathcal{A}$ ? If  $\mathcal{A}$  has an inner MB-representation, say  $\mathcal{A} = S(\mathcal{F})$ , then by Proposition 1 for  $\mathcal{I} = S_0(\mathcal{F})$  we have  $\mathcal{A} = S(\mathcal{A} \setminus \mathcal{I})$ . Is there a canonical ideal  $\mathcal{I}$  with this property? Is such an ideal unique?

To give a positive answer to the first of these questions we need the following fact. Note that, in general,  $\mathcal{F}_2 \subset \mathcal{F}_1$  does not imply  $S(\mathcal{F}_2) \subset S(\mathcal{F}_1)$ . For instance, if  $X = \{0, 1, 2\}$ ,  $\mathcal{F}_2 = \{\{0\}\}$ , and  $\mathcal{F}_1 = \{\{0\}, \{1, 2\}\}$  then  $\{2\} \in S(\mathcal{F}_2) \setminus S(\mathcal{F}_1)$ .

**Lemma 6** *If  $\mathcal{I}_1 \subset \mathcal{I}_2$  are ideals contained in an algebra  $\mathcal{A}$  then we have  $S(\mathcal{A} \setminus \mathcal{I}_2) \subset S(\mathcal{A} \setminus \mathcal{I}_1)$ .*

PROOF. Let  $A \in S(\mathcal{A} \setminus \mathcal{I}_2)$ . To show that  $A \in S(\mathcal{A} \setminus \mathcal{I}_1)$  take a  $P \in \mathcal{A} \setminus \mathcal{I}_1$ . We need to find a  $Q \in \mathcal{A} \setminus \mathcal{I}_1$  for which

$$\text{either } Q \subset P \cap A \text{ or } Q \subset P \setminus A. \quad (1)$$

If  $P \in \mathcal{A} \setminus \mathcal{I}_2$  then clearly there is a  $Q \in \mathcal{A} \setminus \mathcal{I}_2 \subset \mathcal{A} \setminus \mathcal{I}_1$  satisfying (1). So assume that  $P \notin \mathcal{A} \setminus \mathcal{I}_2$ . Then  $P \in \mathcal{I}_2 \setminus \mathcal{I}_1$ . So,  $P \cap A$  and  $P \setminus A$  belong to  $\mathcal{I}_2$  and at least one of them does not belong to  $\mathcal{I}_1$ . This set can be taken as  $Q$ , since  $\mathcal{I}_2 \setminus \mathcal{I}_1 \subset \mathcal{A} \setminus \mathcal{I}_1$ . ■

For an algebra  $\mathcal{A}$  of subsets of  $X$ , the ideal of hereditary sets in  $\mathcal{A}$  is defined as  $\mathcal{H}(\mathcal{A}) = \{A \in \mathcal{A} : \mathcal{P}(A) \subset \mathcal{A}\}$ .

**Proposition 7** *Let  $\mathcal{I}$  be an ideal on a set  $X$ , let  $\mathcal{A}$  be an algebra on  $X$  and assume that  $\mathcal{I} \subset \mathcal{A} = S(\mathcal{A} \setminus \mathcal{I}) \neq \mathcal{P}(X)$ . Then for every ideal  $\mathcal{J}$  such that  $\mathcal{I} \subset \mathcal{J} \subset \mathcal{H}(\mathcal{A})$  we have  $\mathcal{A} = S(\mathcal{A} \setminus \mathcal{J})$ .*

PROOF. Notice that any ideal  $\mathcal{J} \subset \mathcal{A}$  is a proper subset of  $\mathcal{A}$  since  $\mathcal{A} \neq \mathcal{P}(X)$ . It is easy to see that for any such ideal we have  $\mathcal{A} \subset S(\mathcal{A} \setminus \mathcal{J})$ . Indeed, if  $A \in \mathcal{A}$  and  $P \in \mathcal{A} \setminus \mathcal{J}$  then either  $Q = P \setminus A$  belongs to  $\mathcal{A} \setminus \mathcal{J}$  or  $Q = P \cap A$  belongs to  $\mathcal{A} \setminus \mathcal{J}$ . Now, by Lemma 6, we have

$$\mathcal{A} \subset S(\mathcal{A} \setminus \mathcal{H}(\mathcal{A})) \subset S(\mathcal{A} \setminus \mathcal{J}) \subset S(\mathcal{A} \setminus \mathcal{I}) = \mathcal{A}.$$

This finishes the proof. ■

The proposition implies immediately the following corollary, which shows, in particular, that the ideal  $\mathcal{I} = \mathcal{H}(\mathcal{A})$  is canonical ideal in representation  $\mathcal{A} = S(\mathcal{A} \setminus \mathcal{I})$ .

**Corollary 8** *An algebra  $\mathcal{A} \neq \mathcal{P}(X)$  has an inner MB-representation if and only if  $\mathcal{A} = S(\mathcal{A} \setminus \mathcal{H}(\mathcal{A}))$ .*

Notice that Corollary 8 immediately implies [BBC, thm. 13], since conditions (I) and (II) from that theorem say that  $\mathcal{H}(\mathcal{A}) = \mathcal{A} \cap [X]^{<\kappa}$  while (III) says that  $S(\mathcal{A} \setminus \mathcal{H}(\mathcal{A})) \setminus \mathcal{A} \neq \emptyset$ . In particular, Corollary 8 implies easily that the following algebras do not have inner MB-representation:

- The algebra  $\mathcal{B}$  of Borel subset of  $\mathbb{R}$ , since  $S(\mathcal{B} \setminus \mathcal{H}(\mathcal{B})) = S(\mathcal{B} \setminus [\mathbb{R}]^{\leq \omega})$  is a classical Marczewski's algebra. (Compare [BBC, cor. 14].)
- The interval algebra  $\mathcal{A}$  (i.e., generated by all intervals  $[a, b]$ , where  $a, b \in \mathbb{R}$ ), since  $\mathcal{H}(\mathcal{A}) = \{\emptyset\}$  and so  $S(\mathcal{A} \setminus \mathcal{H}(\mathcal{A}))$  is an algebra of subsets of  $\mathbb{R}$  with nowhere dense boundary. (Compare [BBC, prop. 12].)
- The algebra  $\mathcal{A}$  generated by all open intervals  $(a, b)$  ( $a, b \in \mathbb{R}$ ), since  $\mathcal{H}(\mathcal{A}) = [\mathbb{R}]^{<\omega}$  and so  $S(\mathcal{A} \setminus \mathcal{H}(\mathcal{A}))$  is an algebra of subsets of  $\mathbb{R}$  with nowhere dense boundary.

Next, we will address the question of uniqueness of the ideal in the representation  $\mathcal{A} = S(\mathcal{A} \setminus \mathcal{H}(\mathcal{A}))$ . We will start with the following proposition.

**Proposition 9** *Let  $\mathcal{A}$  be an algebra, let  $\mathcal{I} \subset \mathcal{J} \subset \mathcal{A}$  be ideals, and  $Y \in \mathcal{A}$ .*

- (a) *If every  $P \subset Y$  from  $\mathcal{A} \setminus \mathcal{J}$  contains a subset in  $\mathcal{I} \setminus \mathcal{J}$  then  $\mathcal{P}(Y) \subset S(\mathcal{A} \setminus \mathcal{J})$ .*
- (b) *If  $\mathcal{I} \cap \mathcal{P}(Y) = \mathcal{J} \cap \mathcal{P}(Y)$  then  $S(\mathcal{A} \setminus \mathcal{I}) \cap \mathcal{P}(Y) = S(\mathcal{A} \setminus \mathcal{J}) \cap \mathcal{P}(Y)$ .*

PROOF. (a): Let  $A \in \mathcal{P}(Y)$  and take  $P \in \mathcal{A} \setminus \mathcal{J}$ . We need to find a  $Q \in \mathcal{A} \setminus \mathcal{J}$  for which

$$\text{either } Q \subset P \cap A \text{ or } Q \subset P \setminus A.$$

If  $P \in \mathcal{I} \setminus \mathcal{J}$  then either  $P \cap A$  or  $P \setminus A$  belongs to  $\mathcal{I} \setminus \mathcal{J}$ , so we may take this set as a  $Q$ . So, assume that  $P \in \mathcal{A} \setminus \mathcal{I}$  then there is a  $P_0 \in \mathcal{I} \setminus \mathcal{J}$  contained in  $P$ . Thus, as before, either  $P_0 \cap A$  or  $P_0 \setminus A$  belongs to  $\mathcal{I} \setminus \mathcal{J}$  and we may take this set as a  $Q$ .

Part (b) is obvious. ■

For an algebra  $\mathcal{A} \subset \mathcal{P}(X)$  and the ideals  $\mathcal{I}$  and  $\mathcal{J}$  such that  $\mathcal{J} \subset \mathcal{I} \subset \mathcal{A}$  a set  $Y \in \mathcal{A}$  will be called  $\langle \mathcal{I}, \mathcal{J} \rangle$ -special if  $\mathcal{I} \cap \mathcal{P}(X \setminus Y) = \mathcal{J} \cap \mathcal{P}(X \setminus Y)$  and each set  $P \subset Y$  such that  $P \in \mathcal{A} \setminus \mathcal{J}$  has a subset in  $\mathcal{I} \setminus \mathcal{J}$ .

From Proposition 9 we easily derive the following corollary.

**Corollary 10** *Let  $\mathcal{A}$  be an algebra on  $X$  and let  $\mathcal{J} \subset \mathcal{I} \subset \mathcal{A}$  be ideals. If  $Y \in \mathcal{A}$  is an  $\langle \mathcal{I}, \mathcal{J} \rangle$ -special set then*

$$S(\mathcal{A} \setminus \mathcal{J}) = \{C \cup D : C \in \mathcal{P}(Y) \text{ \& } D \in \mathcal{P}(X \setminus Y) \cap S(\mathcal{A} \setminus \mathcal{J})\}.$$

From Proposition 9(a) applied to  $Y = \mathbb{R}$  we obtain immediately the following facts.

- If  $\mathcal{L}$  is the algebra of Lebesgue measurable subsets of  $\mathbb{R}$ ,  $\mathcal{N}$  is the ideal of measure zero sets, and  $\mathcal{N}_0$  is the ideal generated by  $F_\sigma$  sets from  $\mathcal{N}$  then  $S(\mathcal{L} \setminus \mathcal{J}) = \mathcal{P}(\mathbb{R})$  for every ideal  $\mathcal{J}$  contained either in  $\mathcal{N}_0$  or in  $\mathcal{N} \cap [\mathbb{R}]^{<2^\omega}$ .
- If  $\mathcal{B}$  is the algebra of subsets of  $\mathbb{R}$  with the Baire property and  $\mathcal{M}$  is the ideal of meager sets, then  $S(\mathcal{B} \setminus \mathcal{J}) = \mathcal{P}(\mathbb{R})$  for every ideal  $\mathcal{J}$  contained either in  $\mathcal{N}_0$  or in  $\mathcal{M} \cap [\mathbb{R}]^{<2^\omega}$ .

From Corollary 10 we immediately see that, most of the time,  $\mathcal{H}(\mathcal{A})$  is not the only ideal  $I$  for which  $\mathcal{A} = S(\mathcal{A} \setminus I)$ . The easiest way to see it is to notice the following conclusion from Corollary 10.

**Corollary 11** *If  $\mathcal{A}$  is an algebra on  $X$ ,  $\mathcal{J} \subset \mathcal{I} \subset \mathcal{A}$  are ideals,  $\mathcal{A} = S(\mathcal{A} \setminus \mathcal{I})$  and there exists a  $Y \in \mathcal{I}$  such that  $\mathcal{I} \cap \mathcal{P}(X \setminus Y) = \mathcal{J} \cap \mathcal{P}(X \setminus Y)$ , then  $S(\mathcal{A} \setminus \mathcal{I}) = S(\mathcal{A} \setminus \mathcal{J})$ .*

Finally we note that the existence of an  $\langle \mathcal{I}, \mathcal{J} \rangle$ -special set is by no means necessary for the conclusion of Corollary 11.

**Example 12** *There exists an algebra  $\mathcal{A}$  and an ideal  $\mathcal{J} \subsetneq \mathcal{H}(\mathcal{A})$  for which  $\mathcal{A} = S(\mathcal{A} \setminus \mathcal{H}(\mathcal{A})) = S(\mathcal{A} \setminus \mathcal{J})$  while there is no  $\langle \mathcal{H}(\mathcal{A}), \mathcal{J} \rangle$ -special set  $Y \in \mathcal{H}(\mathcal{A})$ .*

PROOF. In the papers [R] and [NR] the authors investigated the family  $\mathcal{D}$  of perfect subsets of  $[\omega]^\omega$ , where  $[\omega]^\omega$  is endowed with the Ellentuck topology, that is, the topology generated by the sets  $[x, y] = \{z \in [\omega]^\omega : x \subset z \subset y\}$ , where  $x \in [\omega]^{<\omega}$  and  $y \in [\omega]^\omega$ . A subset of  $[\omega]^\omega$  is called a *chain* if it consists of sets incomparable with respect to inclusion. A chain is called a *Sorgenfrey chain* if its subspace topology is homeomorphic to the Sorgenfrey topology on  $(0, 1]$ . It is shown in [NR, thm. 3.4] that if  $P \in \mathcal{D}$  does not contain a countable perfect set then  $P$  contains a perfect uncountable Sorgenfrey chain.

Let  $\mathcal{G}$  be the family of all perfect Sorgenfrey chains and let  $\mathcal{A} = S(\mathcal{D})$ . By [NR, thm. 3.5] and [R, cor. 1.10], we have  $\mathcal{A} = S(\mathcal{D}) = S(\mathcal{G})$  and  $\mathcal{J} = S_0(\mathcal{D}) \subsetneq S_0(\mathcal{G}) = \mathcal{H}(\mathcal{A})$ . We will show that

- (a)  $\mathcal{A} = S(\mathcal{A} \setminus \mathcal{J})$ , and



(b)  $\mathcal{A} = S(\mathcal{A} \setminus \mathcal{H}(\mathcal{A}))$ , but

(c) there is no  $\langle \mathcal{H}(\mathcal{A}), \mathcal{J} \rangle$ -special set  $Y \in \mathcal{H}(\mathcal{A})$ .

To prove (a) observe that  $\mathcal{D} \subset S(\mathcal{D})$  since, for any two perfect sets  $P$  and  $Q$ , at least one of the sets  $P \cap Q$ ,  $P \setminus Q$  has a perfect part. Now, from  $\mathcal{D} \subset S(\mathcal{D})$  and  $\mathcal{D} \cap S_0(\mathcal{D}) = \emptyset$  it follows that  $\mathcal{D}$  and  $\mathcal{A} \setminus \mathcal{J} = S(\mathcal{D}) \setminus S_0(\mathcal{D})$  are mutually coinital which, by [BBRW, prop. 1.2], implies (a). The clause (b) results from (a) and Proposition 7.

To prove (c), by way of contradiction assume that there is a  $\langle \mathcal{H}(\mathcal{A}), S_0(\mathcal{D}) \rangle$ -special set  $Y \in \mathcal{H}(\mathcal{A})$ . Then  $\mathcal{H}(\mathcal{A}) \cap \mathcal{P}([\omega]^\omega \setminus Y) = S_0(\mathcal{D}) \cap \mathcal{P}([\omega]^\omega \setminus Y)$ . Since  $\mathcal{H}(\mathcal{A}) = S_0(\mathcal{G})$ , we have

$$S_0(\mathcal{G}) \cap \mathcal{P}([\omega]^\omega \setminus Y) = S_0(\mathcal{D}) \cap \mathcal{P}([\omega]^\omega \setminus Y). \quad (2)$$

Next observe that

(d) each set from  $\mathcal{D} \cap \mathcal{P}([\omega]^\omega \setminus Y)$  contains a set from  $\mathcal{G}$ .

Indeed, let  $D \in \mathcal{D} \cap \mathcal{P}([\omega]^\omega \setminus Y)$ . Since  $\mathcal{D} \subset S(\mathcal{D}) \setminus S_0(\mathcal{D})$ , it follows from  $S(\mathcal{D}) = S(\mathcal{G})$  and (2) that

$$D \in (S(\mathcal{D}) \setminus S_0(\mathcal{D})) \cap \mathcal{P}([\omega]^\omega \setminus Y) = (S(\mathcal{G}) \setminus S_0(\mathcal{G})) \cap \mathcal{P}([\omega]^\omega \setminus Y).$$

Hence by [BBRW, prop 1.1(4)], there is a  $G \in \mathcal{G}$  such that  $G \subset D$  as desired.

Since  $\mathcal{G}$  consists of uncountable sets, from (d) we derive that no countable perfect set in  $[\omega]^\omega$  is contained in  $[\omega]^\omega \setminus Y$ . From [NR] it follows that each nonempty open set in  $[\omega]^\omega$  contains a set from  $\mathcal{G}$ . Thus  $Y$ , which is in  $\mathcal{H}(\mathcal{A}) = S_0(\mathcal{G})$ , has the empty interior. Consequently,  $[\omega]^\omega \setminus Y$  is dense and so, by [R, thm. 1.5], it contains a countable perfect set  $Q$ . However, this contradicts the previous observation.  $\blacksquare$

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